

Low temperature thermodynamics of inverse square spin models in one dimension^{*}

W. Hofstetter^{1,a} and W. Zwerger²

¹ Theoretische Physik III, Universität Augsburg, 86135 Augsburg, Germany

² Sektion Physik, Ludwig-Maximilians-Universität München, Theresienstrasse 37, 80333 München, Germany

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Abstract. We present a field-theoretic renormalization group calculation in two loop order for classical $O(N)$ -models with an inverse square interaction in the vicinity of their lower critical dimensionality one. The magnetic susceptibility at low temperatures is shown to diverge like $T^{-a} \exp(b/T)$ with $a = (N-2)/(N-1)$ and $b = 2\pi^2/(N-1)$. From a comparison with the exactly solvable Haldane-Shastry model we find that the same temperature dependence applies also to ferromagnetic quantum spin chains.

PACS. 75.10.Hk Classical spin models – 75.10.Jm Quantized spin models

1 Introduction

The investigation of long range forces in statistical mechanics has a considerable history, including studies on the basic requirements for a proper thermodynamic limit [1] or on the existence of phase transitions even in one-dimensional systems. For classical spins, it has been known for some time [2] that ferromagnetic models with a pair-interaction decaying like $n^{-(1+\sigma)}$ ($\sigma > 0$ is necessary for a proper thermodynamic limit) exhibit no transition if $\sigma > 1$. Choosing $\sigma < 1$ however, it turns out there is indeed a phase transition even in one dimension both for Ising models [3] as well as for systems with a continuous symmetry [4]. The borderline case $\sigma = 1$ of an inverse square interaction is thus of particular interest. For Ising spins an early conjecture by Thouless [5] that this model exhibits a peculiar continuous transition with a finite jump of the order parameter was finally proven rigorously by Fröhlich and Spencer [6]. In the case of a continuous symmetry like the XY - or Heisenberg-models with inverse square interaction, Simon has shown [7] that no symmetry breaking appears here, suggesting that the borderline interaction exhibiting a Kosterlitz-Thouless type transition is $(\ln n)/n^2$. An alternative proof that the inverse square XY -model in one dimension exhibits no long range order was later given by Simanek [8], using the classical version of the Mermin-Wagner theorem.

A quantitative calculation of the low temperature susceptibility of long range ferromagnetic spin models was given by Kosterlitz [9] within a momentum shell renormalization group (RNG) up to one loop order. In particular the susceptibility was found to diverge exponentially

like $\exp(2\pi^2/(N-1)T)$ in the case $\sigma = 1$ of an inverse square interaction for models with a continuous symmetry, where only the trivial fixed point at $T = 0$ describing a fully ordered ground state exists. Independently, in a brief note, Brézin, Zinn-Justin and Le Guillou [10] presented the results of a two loop field-theoretic RNG of the $O(N)$ -model with interaction proportional to $n^{-(1+\sigma)}$. They determined the critical exponents up to order $(d-\sigma)^2$ in the vicinity of the lower critical dimension $d_c = \sigma$. Here we present a detailed version of the field-theoretic two loop calculation which essentially follows their ideas. However, it uses a somewhat different method to determine the renormalization factors and, moreover, concentrates on the behaviour in the marginal case $d = \sigma = 1$.

The study of one-dimensional classical spin models with long range interactions is interesting not only from a pure statistical mechanics point of view, but is also relevant for actual physical problems. In fact the inverse square Ising-model is closely related to the well-known Kondo-problem of a magnetic impurity locally coupled to the conduction electron spin density [11]. A more recent application of the same model is the two state system with ohmic dissipation, arising *e.g.* in the problem of quantum coherence between macroscopically different states [12, 13]. For the inverse square XY -case it turns out that the problem of strong tunneling in a metallic single-electron-box can be treated by calculating the free energy of the classical spin model in the presence of twisted boundary conditions [14]. It was the latter problem which motivated our present work, whose intention is in fact twofold: first of all, we give a detailed account of the two-loop RNG of the $O(N)$ -model with inverse square interaction which has obviously not been done in the literature so far. In addition to that we discuss the corresponding

^{*} Dedicated to J. Zittartz on the occasion of his 60th birthday

^a e-mail: hofstett@physik.uni-augsburg.de

$S = 1/2$ quantum spin chain. In particular, it is found that the low temperature behaviour of the susceptibility is essentially identical with that for classical spins.

2 RNG in two-loop order

2.1 Model and definitions

We start with a model of classical unit spins $\mathbf{S}(n)$ with N components which are placed on a ring with L sites, thus enforcing periodic boundary conditions. The interaction energy in an external magnetic field H along the N -direction is given by

$$H_{cl} = - \sum_{n \neq n'} [d(n - n')]^{-2} \mathbf{S}(n) \mathbf{S}(n') - H \sum_n S^N(n) \quad (1)$$

with

$$d(n) = \frac{L}{\pi} \sin \frac{\pi|n|}{L} \quad (2)$$

the chord distance between spins which are n sites apart. The associated partition function can then be written as

$$\mathcal{Z} = \int \prod_n d\mathbf{S}(n) \delta(\mathbf{S}^2(n) - 1) \exp\{-S(T, H, \mathbf{S}(n))\}. \quad (3)$$

with an action

$$\mathcal{S} = \frac{1}{2T} \sum_{n \neq n'} \frac{(\mathbf{S}(n) - \mathbf{S}(n'))^2}{(d(n - n'))^2} - \frac{H}{T} \sum_n S^N(n). \quad (4)$$

Here $\int d\mathbf{S}(n) \delta(\mathbf{S}^2(n) - 1)$ denotes an integration over all orientations of a classical N -component unit vector $\mathbf{S}(n)$. As is known from the work cited above, this model exhibits a continuous phase transition to an ordered phase below a critical temperature $T_c \sim \epsilon/(N - 1)$ for noninteger dimensionality $d = 1 + \epsilon$ ($\epsilon > 0$). In order to study the behavior near the lower critical dimensionality one, we can therefore parametrize the spin \mathbf{S} according to a low temperature expansion

$$\mathbf{S}(x) = \left(\boldsymbol{\Pi}(x), \sqrt{1 - \boldsymbol{\Pi}^2(x)} \right). \quad (5)$$

The Π_i ($i = 1, \dots, N - 1$) are Goldstone modes corresponding to the fluctuations around the ordered state with $S_N = 1$ where all spins point in the N -direction. The generating functional is now given by

$$\mathcal{Z} = \int \prod_n \frac{d\boldsymbol{\Pi}(n)}{\sqrt{1 - \boldsymbol{\Pi}^2(n)}} \exp\{-S(T, H, \boldsymbol{\Pi}(n))\}. \quad (6)$$

The term in the integration measure

$$\prod_n \frac{1}{\sqrt{1 - \boldsymbol{\Pi}^2(n)}} = \exp \left\{ -\frac{1}{2} \sum_n \ln(1 - \boldsymbol{\Pi}^2(n)) \right\} \quad (7)$$

is due to the δ -function constraint and ensures the $O(N)$ -symmetry of the model, which appears to be broken by the above parametrization. In a continuum notation $\sum_n \rightarrow a^{-d} \int d^d x$ with a the lattice constant and

$$\prod_n \frac{1}{\sqrt{1 - \boldsymbol{\Pi}^2(n)}} = \exp \left\{ -\frac{1}{2} a^{-d} \int d^d x \ln(1 - \boldsymbol{\Pi}^2(x)) \right\}. \quad (8)$$

In the following, we will work in the dimensional regularization scheme, where

$$a^{-d} = \int^\Lambda \frac{d^d q}{(2\pi)^d} = 0. \quad (9)$$

We can therefore neglect the measure term (see [15]).

In order to generate correlation functions of the $\boldsymbol{\Pi}$ fields, we introduce additional source fields according to

$$\mathcal{S} \mapsto \mathcal{S} - \int dx \mathbf{J}(x) \boldsymbol{\Pi}(x). \quad (10)$$

The quantities which are most conveniently calculated diagrammatically are the irreducible vertex functions $\Gamma^{(n)}$. Their generating functional $\Gamma[\bar{\boldsymbol{\Pi}}, H]$ is formally obtained from $\ln \mathcal{Z}[\mathbf{J}, H]$ by a Legendre transformation with respect to \mathbf{J} where $\bar{\boldsymbol{\Pi}}$ is the corresponding classical field. In our calculation, we will use the two point function

$$\Gamma^{(2)}(x, y) = \frac{\delta^2 \Gamma}{\delta \bar{\boldsymbol{\Pi}}(x) \delta \bar{\boldsymbol{\Pi}}(y)}. \quad (11)$$

As in the case of the non-linear σ -model with next-neighbour coupling [15, 16] it can be shown that the model is renormalizable by introducing two renormalization factors Z_t and Z_π rescaling the temperature and the field $\boldsymbol{\Pi}$. These factors must be adjusted in such a way that the renormalized N -point functions

$$\Gamma_R^{(N)} = Z_\pi^{\frac{N}{2}} Z_t^{-1} \Gamma^{(N)} \quad (12)$$

remain finite in terms of the renormalized parameters

$$t = \kappa^\epsilon Z_t^{-1} T \quad (13)$$

$$h = Z_t^{-1} Z_\pi^{\frac{1}{2}} H. \quad (14)$$

Below we will see that the choice $Z_t = Z_\pi \equiv Z$ with a single independent renormalization constant Z is possible.

2.2 Details of the two-loop-calculation

In order to calculate the Z -factors introduced above to the order t^2 , we first rewrite the action in momentum representation:

$$\mathcal{S} = \frac{1}{T} \int \frac{d^d q}{(2\pi)^d} (|q| + H) \left[\frac{\boldsymbol{\Pi}(q) \boldsymbol{\Pi}(-q)}{2} + \frac{\boldsymbol{\Pi}^2(q) \boldsymbol{\Pi}^2(-q)}{8} + \frac{\boldsymbol{\Pi}^2(q) (\boldsymbol{\Pi}^2)^2(-q)}{16} \right] + O((\boldsymbol{\Pi}^2)^4), \quad (15)$$

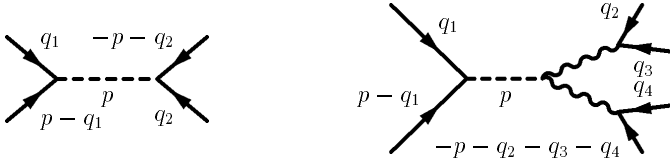


Fig. 1. Diagrammatic representation of the four- and six-point interaction terms.

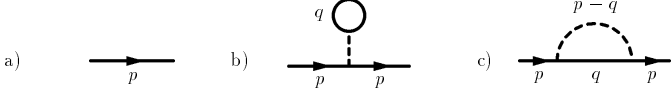


Fig. 2. Free propagator (a) and one-loop contributions to $\Gamma^{(2)}$ (b) and (c).

where we have rescaled $H \mapsto 2\pi H$ and $T \mapsto (2\pi)^d T$. The free propagator corresponding to the quadratic term has the form

$$\begin{aligned} \langle \Pi^\alpha(p) \Pi^\beta(q) \rangle &= \delta_{\alpha\beta} \delta(p+q) G_0^\alpha(p) \\ &= (2\pi)^d \delta_{\alpha\beta} \delta(p+q) \frac{T}{|p|+H}. \end{aligned} \quad (16)$$

with $\alpha, \beta = 1, \dots, N-1$. In addition, \mathcal{S} contains two interaction terms which contribute up to two-loop order. These are shown in Figure 1 and correspond to a four- and a six-point-interaction. We will renormalize the model using the two-point function $\Gamma^{(2)}$. The relevant diagrams up to two-loop order are shown in Figures 2 and 3. In a short hand notation, the two point function in terms of these diagrams is given by

$$\begin{aligned} \Gamma^{(2)}(k) &= a) + b) + c) - d) - e) - f) - g) - h) - i) \\ &\quad + j) + k) + l) + m) + n) + o) + p) + q). \end{aligned} \quad (17)$$

Each one of these diagrams is proportional to T^{L-1} where L is the number of loops. Several of them contain divergences for $\epsilon \rightarrow 0$. In the minimal subtraction scheme, we want to absorb these divergences in the two renormalization constants Z_t and Z_π . After renormalization, $\Gamma_R^{(2)}$ should be an analytic function of the variables h , t and p .

For the action (15) we note that the p -dependent parts of the diagrams are finite. To one-loop order, for instance, only the second diagram (labeled by (c)) depends on the external momentum

$$\int \frac{d^d q}{(2\pi)^d} \frac{1+|p-q|}{1+|q|} = \int \frac{d^d q}{(2\pi)^d} \frac{|p-q|-|q|}{1+|q|}. \quad (18)$$

Evidently this integral is finite for $d < 2$. Because of

$$\Gamma_R^{(2)}(p) = \frac{Z_\pi}{Z_t} \kappa^\epsilon \frac{|p| + Z_t Z_\pi^{-\frac{1}{2}} h}{t} + \text{higher order terms}, \quad (19)$$

the coefficient of $|p|$ in the inverse propagator is renormalized by Z_π/Z_t . As is evident from the one-loop term (18),

this prefactor is already finite. Therefore the choice

$$Z_\pi = Z_t \equiv Z \quad (20)$$

is possible, leaving only one independent renormalization constant. In fact this relation is valid to arbitrary loop order, as was shown by Brézin, Zinn-Justin and Le Guillou [10]. As a consequence, we can take the external momentum p to be zero in the following. One further simplification is due to the fact that in dimensional regularization “empty” integrals of the type $\int d^d q$ are equal to zero. This implies that the contributions of the diagrams in (c), (h), (n) and (p) vanish. Moreover, several contributions turn out to be identical and therefore cancel: this applies to the diagrams (d) and (o) which are given by

$$TH^{2d-1} \iint \frac{d^d q d^d r}{(2\pi)^{2d}} \frac{1+|q-r|}{(1+|q|)(1+|r|)}. \quad (21)$$

Similarly the diagrams (i) and (m) which give a contribution

$$\frac{N-1}{2} TH^{2d-1} \iint \frac{d^d q d^d r}{(2\pi)^{2d}} \frac{1+|q-r|}{(1+|q|)(1+|r|)} \quad (22)$$

are equal and therefore cancel in $\Gamma^{(2)}$. The remaining expression to be evaluated is

$$\Gamma^{(2)}(0, H) = a) + b) - e) - f) - g) + k) + j) + l) + q). \quad (23)$$

In order to extract the singular parts of the integrals, we use the asymptotic expansions

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{1+|q|} = K_d \left(-\frac{1}{\epsilon} + o(\epsilon) \right) \quad (24)$$

and

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{(1+|q|)^2} = K_d (1 + o(\epsilon^2)) \quad (25)$$

where the constant

$$K_d = \frac{2}{(4\pi)^{d/2} \Gamma(d/2)} \quad (26)$$

results from the d -dimensional angular integration. The identities (24, 25) allow to extract the leading singularities in ϵ for all contributions to (23) except (e). By a detailed asymptotic analysis of the corresponding integral, the following singular contributions of this diagram can be obtained:

$$\begin{aligned} \text{(e)} &= \iint \frac{d^d q d^d r}{(2\pi)^{2d}} \frac{1+|r-q|}{(1+|q|)^2(1+|r|)} \\ &= \frac{N-1}{2} TH^{1+2\epsilon} K_d^2 \left(\frac{1}{2\epsilon^2} + \frac{1}{2\epsilon} + o(1) \right). \end{aligned} \quad (27)$$

Neglecting nonsingular parts, the resulting two-point function is therefore finally given by

$$\begin{aligned} \Gamma^{(2)}(0, H) &= \frac{H}{T} - H^{1+\epsilon} \frac{N-1}{2\epsilon} + TH^{1+2\epsilon} \frac{(N-1)(N-2)}{4\epsilon} \\ &\quad + TH^{1+2\epsilon} \frac{3(N-1)^2}{8\epsilon^2} \end{aligned} \quad (28)$$

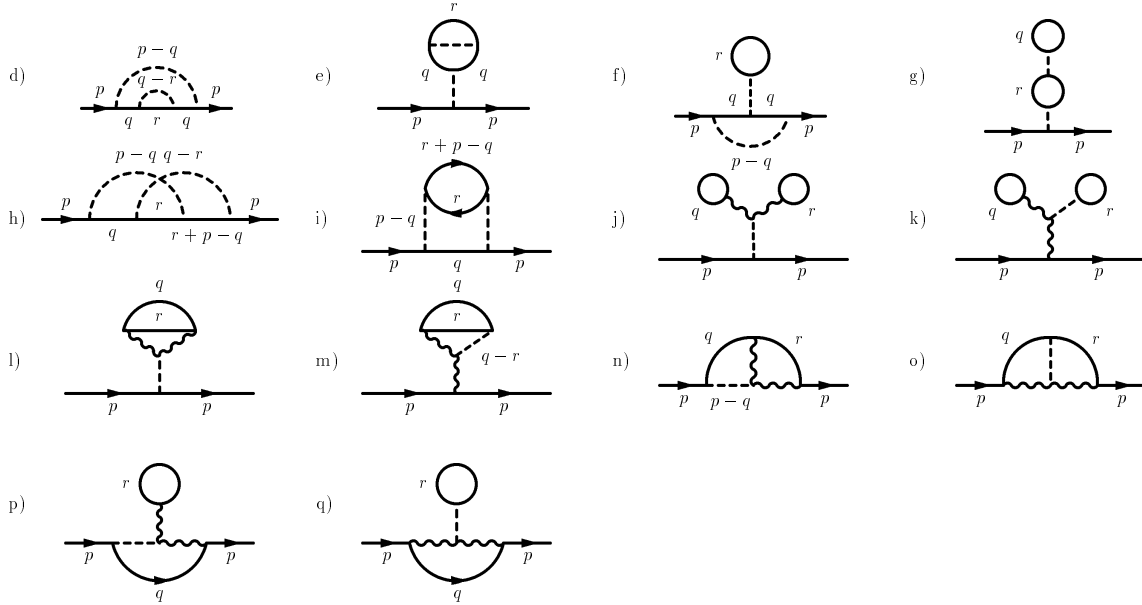


Fig. 3. Two-loop contributions to $\Gamma^{(2)}$.

where we have absorbed the angular integration term K_d in the temperature according to

$$T \mapsto TK_d^{-1}, \quad \Gamma^{(2)} \mapsto \Gamma^{(2)}K_d. \quad (29)$$

At this stage we can now introduce renormalized parameters h and t as defined in (13, 14). Taking $\kappa = 1$ for convenience, we obtain

$$\Gamma_R^{(2)}(0, h) = \frac{Z^{\frac{1}{2}} h}{t} - \frac{N-1}{2\epsilon} Z^{1+\frac{1+\epsilon}{2}} h^{1+\epsilon} + t h^{1+2\epsilon} \frac{(N-1)(N-2)}{4\epsilon} + t h^{1+2\epsilon} \frac{3(N-1)^2}{8\epsilon^2} \quad (30)$$

where terms of order t^2 have been neglected. Using the ansatz

$$Z = 1 + a_1(\epsilon)t + a_2(\epsilon)t^2 + o(t^3) \quad (31)$$

this can be written as

$$\Gamma_R^{(2)}(0, h) = \frac{h}{t} + h \left(\frac{1}{2} a_1 - \frac{N-1}{2\epsilon} \right) + t h \left(\frac{4a_2 - a_1^2}{8} - \frac{(N-1)((3+\epsilon)a_1 - (N-2))}{4\epsilon} + \frac{3(N-1)^2}{8\epsilon^2} \right) + t \ln h \left(\frac{3(N-1)^2}{4\epsilon} - \frac{(N-1)(3+\epsilon)}{2} a_1 \right) \quad (32)$$

where we have neglected terms which are not singular in ϵ . The coefficients a_1 and a_2 are now chosen such as to eliminate the poles of order ϵ^{-1} and ϵ^{-2} . This is achieved by

$$a_1 = \frac{N-1}{\epsilon} \quad (33)$$

and

$$a_2 = \frac{(N-1)^2}{\epsilon^2} + \frac{(N-1)}{2\epsilon}, \quad (34)$$

thus rendering the renormalized two point function finite. The renormalization constant Z is therefore given by

$$Z = 1 + \frac{N-1}{\epsilon} t + \left(\frac{(N-1)^2}{\epsilon^2} + \frac{N-1}{2\epsilon} \right) t^2 + O(t^3). \quad (35)$$

From this result we can now determine the beta function

$$\beta(t) \equiv -\kappa \frac{\partial t}{\partial \kappa} \quad (36)$$

measuring the variation of the renormalized coupling under variation of the momentum scale, keeping the bare parameters T and H fixed. According to (13), we can write

$$\beta(t) = -\epsilon t \left(1 + t \frac{d \ln Z}{dt} \right)^{-1}. \quad (37)$$

Using the result (35) this immediately leads to

$$\beta(t) = -\epsilon t + (N-1)t^2 + (N-1)t^3 + O(t^4). \quad (38)$$

For $\epsilon > 0$, the renormalization flow has a non trivial fixed point corresponding to a phase transition at finite temperature. The associated critical behaviour has been discussed by Brézin *et al.* [10]. Here we are interested in the one-dimensional case, where only the trivial fixed point at $t = 0$ describing the fully ordered ground state exists. During the calculation we have rescaled the coupling T twice. Inserting the original temperature according to $T \rightarrow T/2\pi^2$ we get the following equation for the flow of T under the reduction of the cutoff $\Lambda \rightarrow \Lambda e^{-l}$:

$$\frac{dT}{dl} = (N-1) \frac{T^2}{2\pi^2} + (N-1) \frac{T^3}{(2\pi^2)^2} + O(T^4). \quad (39)$$

Integrating this equation between an initial temperature T very close to zero and a final value $T = O(1)$, the corresponding length rescaling factor behaves like

$$\xi(T) \sim T^{\frac{1}{N-1}} \exp\left(\frac{2\pi^2}{(N-1)T}\right). \quad (40)$$

Obviously $\xi(T)$ plays the role of a ‘‘correlation length’’, however it is important to point out that there can be no exponential decay in models with long range interactions. In fact, using Lieb-Simon-type inequalities, it can be shown that the spin-spin correlation function

$$g(n) = \langle \mathbf{S}(n) \mathbf{S}(0) \rangle \quad (41)$$

asymptotically decays like n^{-2} for arbitrary temperatures [17, 18]. In spite of the absence of an exponential decay, within a standard scaling hypothesis the behavior of $g(n)$ near the critical point at $T = 0$ is described by a universal function $f(n)$ with $f(0) = 1$ and $f(|n| \rightarrow \infty) = f_\infty/n^2$, $f_\infty = O(1)$, such that

$$g(n, T \rightarrow 0) = f\left(\frac{n}{\xi(T)}\right) \quad (42)$$

with $\xi(T)$ given by (40). As a result the magnetic susceptibility

$$\chi = \beta \left(1 + 2 \sum_{n=1}^{\infty} g(n)\right) \quad (43)$$

is finite for any T and scales like

$$\chi(T) \sim T^{-\frac{N-2}{N-1}} \exp\left(\frac{2\pi^2}{(N-1)T}\right) \quad (44)$$

at low temperatures. The two loop calculation thus uniquely determines both the exponent and the T -dependence of the prefactor of the magnetic susceptibility as $T \rightarrow 0$.

3 Discussion

In this work we have calculated the low temperature susceptibility of classical $O(N)$ spin models with an inverse square interaction. Now it is interesting to compare our results with the corresponding ferromagnetic *quantum* spin models. In the case of a Heisenberg model

$$\hat{H} = - \sum_{n \neq n'} J(n-n') \hat{\mathbf{S}}_n \hat{\mathbf{S}}_{n'} - H \sum_n \hat{S}_n^z \quad (45)$$

with interaction $J(n) = d^{-2}(n)$, the exact low temperature susceptibility in the case $S = 1/2$ was calculated by Haldane [19]. It is given by

$$\chi(T) = \frac{1}{4\sqrt{\pi}} T^{-\frac{1}{2}} \exp\left(\frac{\pi^2}{4T}\right) \quad (46)$$

and thus exhibits precisely the same temperature dependence as our result (44) for the classical case with $N = 3$ (note that we have chosen units in which $J = 1$ and our temperature differs from the one in reference [19] by a trivial factor two arising from $\sum_{n \neq n'} = 2 \sum_{n < n'}$). In fact the only difference is in the numerical factor which appears in the exponent. On a qualitative level the exponential divergence of $\chi(T)$ in the case of long range interactions can be understood from a simple spin wave calculation around the perfectly ordered ferromagnetic ground state. For a general spin S , the Hamiltonian (45) leads to a dispersion relation of magnons which has the form

$$\epsilon(k) = 4S \sum_{n>0} J(n) (1 - \cos(kn)) + H. \quad (47)$$

An equilibrium distribution of noninteracting magnons leads to the standard reduction in the dimensionless magnetization

$$m(T, h) = 1 - \frac{1}{S} \int \frac{dk}{2\pi} \frac{1}{e^{\beta\epsilon(k)} - 1}. \quad (48)$$

For the Haldane-Shastry model with $J(n) = d^{-2}(n)$, the magnon dispersion (47) is given by

$$\epsilon(k) = S(2\pi|k| - k^2) + H \quad (49)$$

(see also [19]). The associated reduction in magnetization in the limit $H, T \rightarrow 0$ then behaves like

$$\delta m(T, H) = \frac{T}{2(\pi S)^2} \ln \frac{2\pi S}{H}. \quad (50)$$

The condition $\delta m = b$ with a constant b of order one then defines a field $H(T)$ which qualitatively determines the zero field susceptibility *via*

$$\chi(T) \approx \frac{1}{H(T)} \sim \exp \frac{2b(\pi S)^2}{T}. \quad (51)$$

Choosing $b = 1/2$, this result agrees precisely with the exact solution (46) for $S = 1/2$, while the exponent in the susceptibility of the classical spin model with unit spin is obtained by formally setting $S = 1$. A modified spin wave theory and Schwinger boson techniques have been applied to the Haldane-Shastry model by Nakano and Takahashi [20, 21].

For short range interactions where $J(n)$ decays faster than n^{-3} , the magnon dispersion has the usual form

$$\epsilon(k \rightarrow 0) = Dk^2 + H. \quad (52)$$

It is then straightforward to see that the above spin wave calculation leads to a susceptibility diverging like $\chi(T) \sim S^2/T^2$. This behavior is indeed found from the exact solution of both the nearest neighbor $S = 1/2$ case [22] as well as for the corresponding classical Heisenberg model [23]. We thus expect that for any ferromagnetic spin model with a fully ordered ground state the temperature dependence of the susceptibility at low T is independent of whether the spins are classical or quantum mechanical.

In the case of an XY -model ($N = 2$), the ground state for the $S = 1/2$ chain with inverse square interaction has again a very simple structure, as shown by Haldane [24]. It would thus be interesting to check whether in this case the low temperature susceptibility again diverges like that of the classical model with $N = 2$, *i.e.* $\chi(T) \sim \exp(2\pi^2/T)$.

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